

On Periodic Matrix-Valued Weyl-Titchmarsh Functions

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Abstract

We consider a certain class of Herglotz-Nevanlinna matrix-valued functions which can be realized as the Weyl-Titchmarsh matrix-valued function of some symmetric operator and its self-adjoint extension. New properties of Weyl-Titchmarsh matrix-valued functions as well as a new version of the functional model in such realizations are presented. In the case of periodic Herglotz-Nevanlinna matrix-valued functions we provide a complete characterization of their realizations in terms of the corresponding functional model. We also obtain properties of a symmetric operator and its self-adjoint extension generating periodic Weyl-Titchmarsh matrix-valued function. We study pairs of operators (a symmetric operator and its self-adjoint extension) with constant Weyl-Titchmarsh matrix-valued functions and establish connections between such pairs of operators and representations of the canonical commutation relations for unitary groups of operators in Weyl's form. As a consequence of such an approach we obtain the Stone-von Neumann theorem for two unitary groups of operators satisfying the commutation relations as well as some extension and refinement of the classical functional model for generators of those groups. Our examples include multiplication operators in weighted spaces, first and second order differential operators, as well as the Schrödinger operator with linear potential and its perturbation by bounded periodic potential.

Key words: Weyl-Titchmarsh function, symmetric operator, self-adjoint extension, unitary group.

1 Introduction

In this paper we study a certain class of Herglotz-Nevanlinna matrix-valued functions which can be realized as the Weyl-Titchmarsh matrix-valued function $M_{\mathcal{H},H}(z)$ generated by the densely defined symmetric operator \mathcal{H} and its self-adjoint extension H acting on some Hilbert space \mathfrak{H} [3],[4], [5]. The new properties of these functions as well as a new version of the functional model for the pair (\mathcal{H}, H) in terms of $M_{\mathcal{H},H}(z)$ are obtained. We introduce so called (U, b) -periodic pair of operators (\mathcal{H}, H) , ($U\mathcal{H}U^* = \mathcal{H} - bI$, $UHU^* = H - bI$, U is a unitary operator in \mathfrak{H}) and establish that the Weyl-Titchmarsh matrix-valued function is b -periodic ($M_{\mathcal{H},H}(z+b) = M_{\mathcal{H},H}(z)$) if and only if the corresponding pair of operators (\mathcal{H}, H) generating this matrix-valued function is (U, b) -periodic. It is shown that any Weyl-Titchmarsh function $M_{\mathcal{H},H}(z)$ corresponding to symmetric operator \mathcal{H} with the defect indices $(1, 1)$ which admits quasi-hermitian extension \mathcal{H}_v without spectrum is always $\pi/\text{tr}(\Im \mathcal{H}_v^{-1})$ -periodic. Each (U, b) -periodic symmetric operator \mathcal{H} is associated with a group Γ of transformations of the set $U(m)$ of all $m \times m$ unitary matrices into itself. It turned out that the group Γ is cyclic if and only if an operator \mathcal{H} admits periodic extension. We consider pair of operators (\mathcal{H}, H) with the constant Weyl-Titchmarsh matrix-valued functions and find out connections between such type of pairs and representations of the canonical commutation relations for unitary groups of operators in Weyl's form. As a consequence of such approach we obtain the Stone-von Neumann theorem [8] for two unitary groups of operators satisfying the commutation relations as well as some extension and refinement of the classical functional model for generators of those groups. The examples of the Schrödinger operator with linear potential and its perturbation by bounded periodic function and are considered.

2 The Weyl-Titchmarsh function.

Let \mathfrak{H} be a Hilbert space, \mathcal{H} be a prime symmetric operator in \mathfrak{H} , that is \mathfrak{H} does not contain a proper subspace that reduces \mathcal{H} , and in which \mathcal{H} induces self-adjoint operator. Let $\mathfrak{D}(\mathcal{H})$ denotes the domain of \mathcal{H} . We assume that defect index of \mathcal{H} is (m, m) , $m < \infty$. It means that for any non-real z defect subspace $\mathfrak{N}_z = [(\mathcal{H} - \bar{z}I)\mathfrak{D}(\mathcal{H})]^\perp$ has dimension m . Let H be a self-adjoint extension of \mathcal{H} in \mathfrak{H} (an orthogonal extension) with domain $\mathfrak{D}(H)$. The Weyl-Titchmarsh function of the pair (\mathcal{H}, H) , $M_{\mathcal{H},H}(z)$, is an operator-valued function whose values are operators on m -dimensional space \mathfrak{N}_i . $M_{\mathcal{H},H}(z)$ is defined on the resolvent set $\rho(H)$ of the operator H by

$$M_{\mathcal{H},H}(z) = P_+(zH + I)(H - zI)^{-1}|_{\mathfrak{N}_i} \quad (1)$$

where P_+ is the orthogonal projection from \mathfrak{H} onto \mathfrak{N}_i . From spectral representation of H it follows that $M_{\mathcal{H},H}(z)$ can be written as

$$M_{\mathcal{H},H}(z) = \int_{\mathbb{R}} \frac{\lambda z + 1}{\lambda - z} d\sigma(\lambda). \quad (2)$$

Values of a nondecreasing function $\sigma(\lambda)$ are operators on \mathfrak{N}_i , and it is defined by $\sigma(\lambda) = P_+ E(\lambda)|_{\mathfrak{N}_i}$, where $E(\lambda)$ is the resolution of identity associated with H . We normalize $E(\lambda)$ by condition $E(\lambda) = 1/2(E(\lambda + 0) + E(\lambda - 0))$. It is evidently that $M_{\mathcal{H},H}(z)$ is analytic on $\rho(H)$, particularly, for $\Im z \neq 0$, and from (2) it follows that $\Im M_{\mathcal{H},H}(z) \geq 0$ for $z \in \mathbb{C}_+$. Therefore, $M_{\mathcal{H},H}(z)$ belongs to the Herglotz-Nevalinna class.

Function σ has the following properties:

$$\int_{\mathbb{R}} d\sigma(\lambda) = I_{\mathfrak{N}_i}; \quad (3)$$

$$\int_{\mathbb{R}} (1 + \lambda^2)(d\sigma(\lambda)h, h) = \infty \quad \forall h \in \mathfrak{N}_i, \quad (4)$$

and $\sigma(\lambda) = 1/2(\sigma(\lambda + 0) + \sigma(\lambda - 0))$. Condition (3) is obvious, condition (4) follows from the fact, that according to von Neumann's formulas, for vector $h \in \mathfrak{N}_i$, $h \notin \mathfrak{D}(H)$. Condition (3) provides normalization condition for the Weyl-Titchmarsh function: $M_{\mathcal{H},H}(i) = iI_{\mathfrak{N}_i}$. From condition (4) it follows that points of growth of σ form a noncompact set.

Selecting an orthonormal basis in \mathfrak{N}_i we can identify the space \mathfrak{N}_i with \mathbb{C}^m , and regard $M_{\mathcal{H},H}(z)$ and $\sigma(\lambda)$ as operators on \mathbb{C}^m . Matrices of these operators with respect to the selected basis are also denoted by $M_{\mathcal{H},H}(z)$ and $\sigma(\lambda)$.

Important property of the Weyl-Titchmarsh functions is given by the following theorem.

Theorem 1 . *Let \mathcal{H} and $\tilde{\mathcal{H}}$ be prime symmetric operators with equal defect numbers in Hilbert spaces \mathfrak{H} and $\tilde{\mathfrak{H}}$ respectively, and H and \tilde{H} be their self-adjoint extensions. Suppose that there is the unitary operator $W : \mathfrak{H} \rightarrow \tilde{\mathfrak{H}}$ such that $W\mathcal{H} = \tilde{\mathcal{H}}W$ and $WH = \tilde{H}W$. Then there is a unitary operator $W_0 : \mathfrak{N}_i \rightarrow \tilde{\mathfrak{N}}_i$ such that $W_0 M_{\mathcal{H},H}(z) = M_{\tilde{\mathcal{H}},\tilde{H}}(z)W_0$.*

Proof. From the assumptions of the Theorem it follows that $WE(\lambda) = \tilde{E}(\lambda)W$, where $E(\lambda)$ and $\tilde{E}(\lambda)$ are the resolutions of identity, associated with H and \tilde{H} respectively. It is also obvious that $W\mathfrak{N}_z = \tilde{\mathfrak{N}}_z$, and $WP_+ = \tilde{P}_+W$.

Put $W_0 = W|_{\mathfrak{N}_i}$. Then W_0 is the unitary operator from \mathfrak{N}_i onto $\tilde{\mathfrak{N}}_i$, $W_0^* =$

$W^*|\tilde{\mathfrak{N}}_i$. For any $f \in \mathfrak{N}_i$ and $\tilde{g} \in \tilde{\mathfrak{N}}_i$ we have

$$\begin{aligned} (W_0 M_{\mathcal{H}, H}(z)f, \tilde{g}) &= (W M_{\mathcal{H}, H}(z)f, \tilde{g}) = (M_{\mathcal{H}, H}(z)f, W^* \tilde{g}) = \\ &= \int_{\mathbb{R}} \frac{\lambda z + 1}{\lambda - z} d(P_+ E(\lambda)f, W^* \tilde{g}) = \int_{\mathbb{R}} \frac{\lambda z + 1}{\lambda - z} d(W P_+ E(\lambda)f, \tilde{g}) = \\ &= \int_{\mathbb{R}} \frac{\lambda z + 1}{\lambda - z} d(\tilde{P}_+ \tilde{E}(\lambda)Wf, \tilde{g}) = (M_{\tilde{\mathcal{H}}, \tilde{H}}(z)W_0 f, \tilde{g}). \end{aligned}$$

These equalities show that W_0 possesses desired property.

If $\{e_j\}_{j=1}^m$ is an arbitrary orthonormal basis in \mathfrak{N}_i , then $\{W_0 e_j\}$ is the orthonormal basis in $\tilde{\mathfrak{N}}_i$. With respect to these bases matrices of $M_{\mathcal{H}, H}(z)$ and $M_{\tilde{\mathcal{H}}, \tilde{H}}(z)$ are equal. Therefore, the Theorem 1 can be reformulated as following:

If pairs (\mathcal{H}, H) and $(\tilde{\mathcal{H}}, \tilde{H})$ are unitarily equivalent, then there are bases with respect to which matrices of their Weyl- Titchmarsh functions are equal. The next Theorem is the statement about realization. It provides the functional model of the pair with prescribed Weyl-Titchmarsh function.

Theorem 2 *Let $F(z)$ be a function whose values are linear operators on m -dimensional space \mathfrak{N} , and which admits integral representation*

$$F(z) = \int_{-\infty}^{\infty} \frac{\lambda z + 1}{\lambda - z} d\sigma(\lambda)$$

where $\sigma(\lambda)$ is a nondecreasing function with values on the set of linear operators on \mathfrak{N} , and which satisfies (3) and (4). Then there are Hilbert space $\tilde{\mathfrak{H}}$, prime symmetric operator $\tilde{\mathcal{H}}$ with defect index (m, m) , and its self-adjoint extension \tilde{H} in $\tilde{\mathfrak{H}}$, such that $F(z) = M_{\tilde{\mathcal{H}}, \tilde{H}}(z)$. If $(\hat{\mathfrak{H}}, \hat{\mathcal{H}}, \hat{H})$ is another realization of F , then there is a unitary operator $\Psi : \tilde{\mathfrak{H}} \rightarrow \hat{\mathfrak{H}}$ such that $\Psi \tilde{\mathcal{H}} = \hat{\mathcal{H}} \Psi$, and $\Psi \tilde{H} = \hat{H} \Psi$.

Proof. Since $\sigma(\lambda)$ is nondecreasing operator-function and satisfies (3), it is the generalized resolution of identity which acts in \mathfrak{N} . According to the theorem of M.A.Najmark (see, for example [1]) there exist a Hilbert space $\tilde{\mathfrak{H}}$ which contains \mathfrak{N} as a subspace and the orthogonal resolution of identity $\tilde{E}(\lambda)$, such that for any Borel set $\Delta \in \mathcal{B}(\mathbb{R})$ ($\mathcal{B}(\mathbb{R})$ is the Borel field of \mathbb{R}) $\sigma(\Delta) = P \tilde{E}(\Delta)|\mathfrak{N}$, where P is the orthogonal projection from $\tilde{\mathfrak{H}}$ onto \mathfrak{N} . The space $\tilde{\mathfrak{H}}$ can be selected minimal in that sense that $c.l.h.\{\tilde{E}(\Delta)h|\Delta \in \mathcal{B}(\mathbb{R}), h \in \mathfrak{N}\} = \tilde{\mathfrak{H}}$, where *c.l.h.* means closed linear hall. The orthogonal resolution of identity $\tilde{E}(\lambda)$ defines the self-adjoint operator \tilde{H} in $\tilde{\mathfrak{H}}$. Under minimality condition the Hilbert space $\tilde{\mathfrak{H}}$ and the operator \tilde{H} are defined uniquely up to unitary equivalence.

In our situation this construction gives Hilbert space $\tilde{\mathfrak{H}} = L^2(\mathbb{R}, \mathfrak{N}, d\sigma)$. Elements of $\tilde{\mathfrak{H}}$ are measurable functions $f(\lambda)$, $\lambda \in \mathbb{R}$ with values in \mathfrak{N} such that

$$\int_{\mathbb{R}} (d\sigma(\lambda)f(\lambda), f(\lambda))_{\mathfrak{N}} < \infty.$$

The space \mathfrak{N} is identified with subspace of $L^2(\mathbb{R}, \mathfrak{N}, d\sigma)$ which consists of constant functions. The orthogonal resolution of identity \tilde{E} is defined as $\tilde{E}(\Delta)f(\lambda) = \chi_{\Delta}(\lambda)f(\lambda)$, where χ_{Δ} is the indicator function of the set Δ .

The self-adjoint operator \tilde{H} defined as follows:

$$\mathfrak{D}(\tilde{H}) = \{f \in \tilde{\mathfrak{H}} \mid \int_{\mathbb{R}} (1 + \lambda^2)(d\sigma(\lambda)f(\lambda), f(\lambda))_{\mathfrak{N}} < \infty\}, \quad (5)$$

$$(\tilde{H}f)(\lambda) = \lambda f(\lambda), \quad f \in \mathfrak{D}(\tilde{H}). \quad (6)$$

From (4) it follows that \tilde{H} is unbounded operator.

Put

$$\mathfrak{D}(\tilde{\mathcal{H}}) = \{f \in \mathfrak{D}(\tilde{H}) \mid \int_{\mathbb{R}} (\lambda + i)d\sigma(\lambda)f(\lambda) = 0\}, \quad (7)$$

and

$$(\tilde{\mathcal{H}}f)(\lambda) = \lambda f(\lambda), \quad \lambda \in \mathfrak{D}(\tilde{\mathcal{H}}). \quad (8)$$

$\mathfrak{D}(\tilde{\mathcal{H}})$ is linear manifold, dense in $\tilde{\mathfrak{H}}$ (this fact follows from (4)), and $(\tilde{\mathcal{H}}f, g) = (f, \tilde{\mathcal{H}}g)$ for $f, g \in \mathfrak{D}(\tilde{\mathcal{H}})$. Thus, $\tilde{\mathcal{H}}$ is a symmetric operator. Moreover, condition (7) implies, that $\mathfrak{N} = [(\tilde{\mathcal{H}} + iI)\mathfrak{D}(\tilde{\mathcal{H}})]^{\perp} = \mathfrak{N}_i$. Indeed, for $f \in L^2(\mathbb{R}, \mathfrak{N}, d\sigma)$ put $f_0 = \int d\sigma(\lambda)f$. Then we have $f = (\lambda + i)g + h$, where $g = (f - f_0)/(\lambda + i) \in \mathfrak{D}(\tilde{\mathcal{H}})$, $h = f_0 \perp (\lambda + i)g$. Therefore, one of the defect numbers of $\tilde{\mathcal{H}}$ is m . It is easily seen, that $\mathfrak{N}_{-i} = \{\frac{\lambda - i}{\lambda + i}\xi \mid \xi \in \mathfrak{N}\}$, which means that $\dim \mathfrak{N}_{-i} = m$, and defect index of $\tilde{\mathcal{H}}$ is (m, m) . In general, for arbitrary nonreal z the defect subspace $\mathfrak{N}_z = \{\frac{\lambda - i}{\lambda - z}\xi \mid \xi \in \mathfrak{N}\}$.

The Weyl-Titchmarsh function for the pair $(\tilde{\mathcal{H}}, \tilde{H})$ is

$$M_{\tilde{\mathcal{H}}, \tilde{H}} = P_+(zH + I)(H - zI)^{-1}|_{\mathfrak{N}_i} = \int_{\mathbb{R}} \frac{z\lambda + 1}{\lambda - z} d\sigma(\lambda)$$

coincides with the given function F . Uniqueness of this realization (up to unitary equivalence) is provided by the Najmark's theorem.

Combining results of the Theorems 1 and 2 we obtain the following statement (see [4], [5]).

Corollary 1 . *Let \mathcal{H} be a prime symmetric operator on a Hilbert space \mathfrak{H} with index of defect (m, m) ($m < \infty$), and let H be a self-adjoint extension*

of \mathcal{H} in \mathfrak{H} . Let $M_{\mathcal{H},H}(z)$ be the Weyl-Titchmarsh function of the pair (\mathcal{H}, H) . Let $(\tilde{\mathfrak{H}}, \tilde{\mathcal{H}}, \tilde{H})$ be the realization of $M_{\mathcal{H},H}$ described in Theorem 2. Then there is the unitary operator $\Phi : \mathfrak{H} \rightarrow \tilde{\mathfrak{H}}$ such that

$$\tilde{\mathcal{H}} = \Phi \mathcal{H} \Phi^*, \quad (9)$$

and

$$\tilde{H} = \Phi H \Phi^*. \quad (10)$$

Let U be a unitary operator on \mathfrak{H} , and $\tilde{U} = \Phi U \Phi^*$ be its representation in the model space $\tilde{\mathfrak{H}}$. We say that the operator U is of **shift-type** (s-type) operator if for $f \in \tilde{\mathfrak{H}}$

$$(\tilde{U}f)(\lambda) = D \frac{\lambda - i}{\lambda - i - b} f(\lambda - b), \quad (11)$$

where D is a unitary operator on \mathfrak{N} which commutes with $\sigma(\lambda)$, and b is a real number.

Often it is more convenient to use the following realization of F (see [4],[5]). Put

$$d\tau(\lambda) = (1 + \lambda^2)d\sigma(\lambda). \quad (12)$$

Then

$$F(z) = \int_{-\infty}^{\infty} \left[\frac{1}{\lambda - z} - \frac{\lambda}{1 + \lambda^2} \right] d\tau(\lambda) \quad (13)$$

The mapping $W : L^2(\mathbb{R}, \mathfrak{N}, d\sigma) \rightarrow L^2(\mathbb{R}, \mathfrak{N}, d\tau)$, $(Wf)(\lambda) = f(\lambda)/(\lambda - i)$ is a unitary one. For the self-adjoint operator $\hat{H} = WHW^*$ we have

$$\mathfrak{D}(\hat{H}) = \{f \in L^2(\mathbb{R}, \mathfrak{N}, d\tau) \mid \int_{\mathbb{R}} (1 + \lambda^2)(d\tau(\lambda)f(\lambda), f(\lambda))_{\mathfrak{N}} < \infty\},$$

and $\hat{H}f(\lambda) = \lambda f(\lambda)$.

For symmetric operator $\hat{\mathcal{H}} = W\mathcal{H}W^*$

(i) $\mathfrak{D}(\hat{\mathcal{H}}) = \{f \in \mathfrak{D}(\hat{H}) \mid \int_{\mathbb{R}} f(\lambda)d\tau(\lambda) = 0\};$

(ii) $(\hat{\mathcal{H}}f)(\lambda) = \lambda f(\lambda);$

$$\mathfrak{N}_z = \left\{ \frac{1}{\lambda - z} \xi \mid \xi \in \mathfrak{N} \right\}.$$

In such representation the s-type unitary operator U acts as $(\tilde{U}f)(\lambda) = Df(\lambda - b)$.

Some additional properties of the Weyl-Titchmarsh functions and their applications can be found in [4], [5].

3 Periodic Operators.

Let \mathcal{H} be a prime symmetric operator with index of defect (m, m) , $m < \infty$ and H be its orthogonal self-adjoint extension. In this section we study pairs (\mathcal{H}, H) for which the Weyl-Titchmarsh function is b -periodic, that is

$$M_{\mathcal{H}, H}(z) = M_{\mathcal{H}, H}(z + b), \quad (14)$$

where b is some real number.

We start from the following lemma.

Lemma 1 . *Let $F(z)$ be a function whose values are linear operators on m -dimensional space \mathfrak{N} , and which admits integral representation*

$$F(z) = \int_{-\infty}^{\infty} \frac{\lambda z + 1}{\lambda - z} d\sigma(\lambda) = zI_{\mathfrak{N}} + (1 + z^2) \int_{-\infty}^{\infty} \frac{1}{\lambda - z} d\sigma(\lambda),$$

where $\sigma(\lambda)$ is a nondecreasing function with values on the set of linear operators on \mathfrak{N} which satisfies conditions (3) and (4). The function $F(z)$ is b -periodic, if and only if

$$\tau(\Delta + b) = \tau(\Delta) \quad (15)$$

for any $\Delta \in \mathcal{B}(\mathbb{R})$, where τ is defined by (12).

Proof. In order to prove the Lemma we need the following generalization of the Stieltjes inversion formula. This generalization due to M.Livsic (see [6], Lemma 2.1):

Let $\sigma(\lambda) = 1/2(\sigma(\lambda + 0) + \sigma(\lambda - 0))$ ($-\infty < \lambda < \infty$) be some function of bounded variation on each finite interval, such that the integral

$$\Phi(z) = \int_{-\infty}^{\infty} \frac{d\sigma(\lambda)}{\lambda - z}$$

converges absolutely.

Let $\varphi(\lambda)$ be some function analytic on the closed interval $\Delta = [\alpha, \beta]$.

Denote by Δ_{ϵ} the broken path of integration consisting of directed segment $[\alpha - i\epsilon, \beta - i\epsilon]$ and antiparallel segment $[\beta + i\epsilon, \alpha + i\epsilon]$.

Then

$$\lim_{\epsilon \rightarrow 0} \frac{1}{2\pi i} \int_{\Delta_\epsilon} \varphi(z) \Phi(z) dz = - \int_{\alpha}^{\beta} \varphi(\lambda) d\sigma(\lambda).$$

Fix and orthonormal basis $\{e_j\}_{j=1}^m$ in the space \mathfrak{N} . Condition of b -periodicity of the function $F(z)$ gives

$$b\delta_{jk} + (1 + (z + b)^2) \int_{-\infty}^{\infty} \frac{1}{\lambda - b - z} d\sigma_{jk}(\lambda) = (1 + z^2) \int_{-\infty}^{\infty} \frac{1}{\lambda - z} d\sigma_{jk}(\lambda), \quad (16)$$

$\Im z \neq 0$, and $\sigma_{jk}(\lambda) = (\sigma(\lambda)e_k, e_j)$. Since $\dim \mathfrak{N} = m < \infty$, variations of all functions σ_{jk} , $j, k = 1, 2, \dots, m$ are uniformly bounded and (15) follows from the Livsic's lemma. Indeed, evaluating the integral of both sides of (16) along Δ_ϵ and then taking the limit as $\epsilon \rightarrow 0$ we obtain

$$\int_{\alpha}^{\beta} [1 + (\lambda + b)^2] d\sigma(\lambda + b) = \int_{\alpha}^{\beta} (1 + \lambda^2) d\sigma(\lambda),$$

which is (15).

Suppose now that (15) is fulfilled. Then we have

$$F(z + b) - F(z) = \int_{\mathbb{R}} \left[\frac{1}{\lambda - z - b} - \frac{1}{\lambda - z} \right] d\tau(\lambda) = c,$$

$c = \int_{\mathbb{R}} [\lambda/(1 + \lambda^2) - (\lambda + b)/(1 + (\lambda + b)^2)] d\tau(\lambda)$, and the integrals converge absolutely. We assume for simplicity that $m = 1$ (for case $m < \infty$ the proof can be done by componentwise arguments). Consider difference

$$|F(iy + b) - F(iy)| = \left| \int_{\mathbb{R}} \left[\frac{1}{\lambda - iy - b} - \frac{1}{\lambda - iy} \right] d\tau(\lambda) \right| \leq b \int_{\mathbb{R}} \frac{d\tau(\lambda)}{\sqrt{\lambda^2 + y^2} \sqrt{(\lambda - b)^2 + y^2}}.$$

For large y we have $1/(\sqrt{\lambda^2 + y^2} \sqrt{(\lambda - b)^2 + y^2}) \leq 1/(\sqrt{\lambda^2 + 1} \sqrt{(\lambda - b)^2 + 1})$, therefore, there is $A > 0$ such that

$$\left(\int_{-\infty}^{-A} + \int_A^{\infty} \right) \frac{d\tau(\lambda)}{\sqrt{\lambda^2 + y^2} \sqrt{(\lambda - b)^2 + y^2}} < \frac{\epsilon}{2}$$

for any $\epsilon > 0$ uniformly with respect to y . From the other side,

$$\int_{-A}^A \frac{d\tau(\lambda)}{\sqrt{\lambda^2 + y^2} \sqrt{(\lambda - b)^2 + y^2}} \leq \frac{1 + A^2}{y^2} < \frac{\epsilon}{2}$$

for y large enough (we used the fact that $\int_{\mathbb{R}} d\tau(\lambda)/(1 + \lambda^2) = 1$). We have proved that the constant $c = 0$, and $F(z + b) = F(z)$.

The Lemma is proved now.

Definition. An operator T acting on a Hilbert space \mathfrak{H} with domain $\mathfrak{D}(T)$ is said to be (U, b) -**periodic**, if there is a unitary operator U such that

$$U\mathfrak{D}(T) \subset \mathfrak{D}(T), \quad (17)$$

$$UTU^* = T - bI \quad (18)$$

for some number b .

Of course, periodic operator cannot be bounded. One can easily see that if the operator T^* exists, then it is (U, \bar{b}) -periodic.

We say that prime symmetric operator \mathcal{H} in \mathfrak{H} and its self-adjoint extension H form a (U, b) -**periodic pair**, if conditions (17) and (18) are fulfilled for both of them (with the same unitary operator U).

It is evidently, that if \mathcal{H} is a (U, b) -periodic periodic operator, and \mathfrak{N}_z is a defect subspace of \mathcal{H} , then $U\mathfrak{N}_z = \mathfrak{N}_{z+b}$.

Proposition 1 . *Let \mathcal{H} be a prime symmetric operator, $H \supset \mathcal{H}$ be its selfadjoint extension such that the pair (\mathcal{H}, H) is (U, b) -periodic and (V, b) -periodic. Then the unitary operator $W = V^*U$ has following properties:*

- (1) W commutes with H ;
- (2) each defect subspace \mathfrak{N}_z reduces W ;
- (3) if \mathcal{H} has defect index (m, m) , $m < \infty$, then the spectrum of W consists of finite number of eigenvalues; number of distinct eigenvalues not greater than m .

Indeed, properties 1 and 2 follow directly from the definitions above. The property 3 follows from the fact that the operator W commutes with the resolution of identity $E(\lambda)$ associated with H , *c.l.h.* $\{E(\Delta)\mathfrak{N} | \Delta \in \mathcal{B}(\mathbb{R})\} = \mathfrak{H}$, where \mathfrak{N} is a defect subspace of \mathcal{H} , and the spectrum of $W|_{\mathfrak{N}}$ consists of finite numebrs of eigenvalues.

Theorem 3 *Let \mathcal{H} be a prime symmetric operator on a Hilbert space \mathfrak{H} with defect index (m, m) , $(m < \infty)$, and let H be its self-adjoint extension in \mathfrak{H} .*

Then the following conditions are equivalent:

- (1) The Weyl-Titchmarsh function $M_{\mathcal{H},H}(z)$ of the pair (\mathcal{H}, H) is b -periodic;
- (2) The pair (\mathcal{H}, H) is (U, b) -periodic, where U is an s -type operator.

Proof. Let pair (\mathcal{H}, H) has b -periodic Weyl-Titchmarsh function. Let $(\tilde{\mathfrak{H}}, \tilde{\mathcal{H}}, \tilde{H})$ be the realization of $(\mathfrak{H}, \mathcal{H}, H)$, described in the Theorem 2. According to the Lemma 1 the function $\sigma(\lambda)$ satisfies the periodicity condition $(1 + (\lambda + b)^2)d\sigma(\lambda + b) = (1 + \lambda^2)d\sigma(\lambda)$. On the space $\tilde{\mathfrak{H}} = L^2(\mathbb{R}, \mathfrak{N}_i, d\sigma)$ consider the operator $\tilde{U} : f \rightarrow \tilde{U}f$ defined by

$$(\tilde{U}f)(\lambda) = \frac{\lambda - i}{\lambda - b - i} f(\lambda - b). \quad (19)$$

Operator \tilde{U} is a unitary operator in $L^2(\mathbb{R}, \mathfrak{N}_i, d\sigma)$. Indeed,

$$\begin{aligned} (\tilde{U}f, \tilde{U}f) &= \int_{-\infty}^{\infty} \frac{\lambda^2 + 1}{1 + (\lambda - b)^2} (d\sigma(\lambda) f(\lambda - b), f(\lambda - b)) = \\ &= \int_{-\infty}^{\infty} \frac{1 + (\lambda - b)^2}{1 + (\lambda - b)^2} d(\sigma(\lambda - b) f(\lambda - b), f(\lambda - b)) = (f, f). \end{aligned}$$

The domain of the operator $\tilde{\mathcal{H}}$ is invariant under \tilde{U} . For $f \in \mathfrak{D}(\tilde{\mathcal{H}})$, that is $\int_{\mathbb{R}} (\lambda + i) d\sigma(\lambda) f(\lambda) = 0$, we have

$$\begin{aligned} \int_{-\infty}^{\infty} (\lambda + i) d\sigma(\lambda) (\tilde{U}f)(\lambda) &= \int_{-\infty}^{\infty} \frac{\lambda^2 + 1}{\lambda - i - b} d\sigma(\lambda) f(\lambda - b) = \\ &= \int_{-\infty}^{\infty} \frac{1 + (\lambda - b)^2}{\lambda - b - i} d\sigma(\lambda - b) f(\lambda - b) = \int_{-\infty}^{\infty} (\lambda + i) d\sigma(\lambda) f(\lambda) = 0. \end{aligned}$$

It is obvious that if $f \in \mathfrak{D}(\tilde{H})$, then $\tilde{U}\tilde{H}f = (\tilde{H} - bI)\tilde{U}f$. Therefore, $(\tilde{\mathcal{H}}, \tilde{H})$ is the (\tilde{U}, b) -periodic pair. Therefore, the pair (\mathcal{H}, H) is the (U, b) -periodic one, and U is the s -type operator.

Conversely, let (\mathcal{H}, H) be a (U, b) -periodic pair, with operator U of s -type. Therefore, in the realization $(\tilde{\mathfrak{H}}, \tilde{\mathcal{H}}, \tilde{H})$ the pair $(\tilde{\mathcal{H}}, \tilde{H})$ is (\tilde{U}, b) -periodic, with \tilde{U} of the form (11). From the equation $\tilde{U}\tilde{H}\tilde{U}^* = \tilde{H} - bI$ it follows that the resolution of identity $\tilde{E}(\lambda)$ of the operator \tilde{H} satisfies the condition

$$\tilde{U}\tilde{E}(\lambda)\tilde{U}^* = \tilde{E}(\lambda + b). \quad (20)$$

If $\hat{\mathfrak{N}}_i$ is the defect subspace of the operator $\tilde{U}\tilde{H}\tilde{U}^*$, then $\hat{\mathfrak{N}}_i = \mathfrak{N}_{i+b}$. Let $\{e_j\}$

be an orthonormal basis in \mathfrak{N} . Then $\tilde{U}e_j = \frac{\lambda - i}{\lambda - i - b}De_j$, $j = 1, 2, \dots, m$ is the orthonormal basis in $\hat{\mathfrak{N}}_i = \mathfrak{N}_{i+b}$. Now the Theorem 1 gives

$$\begin{aligned}\sigma_{jk}(\lambda) &= (\tilde{E}(\lambda)e_k, e_j) = (\tilde{E}(\lambda + b)\tilde{U}e_k, \tilde{U}e_j) = \\ &= \int_{-\infty}^{\lambda+b} \frac{1 + s^2}{1 + (s - b)^2} d\sigma(s),\end{aligned}$$

from which we get $(1 + \lambda^2)d\sigma(\lambda) = (1 + (\lambda + b)^2)d\sigma(\lambda + b)$.

Therefore, the function σ satisfies the condition of the Lemma 1, and $M_{\mathcal{H}, H}(z)$ is the b -periodic function. The Theorem is proved.

Remark. It can be proved, that if (\mathcal{H}, H) is a (U, b) -periodic pair, where index of defect of \mathcal{H} is $(1, 1)$, then the unitary operator U is necessarily of s-type.

Lemma 2 . *Let \mathcal{H} be a (U, b) -periodic prime symmetric operator with finite and equal defect numbers, and let (\mathcal{H}, H_0) is a (U, b) -periodic pair. Define operator functions $\mathcal{A}(z)$ and $\mathcal{B}(z)$ by the equations*

$$\mathcal{A}(z) = \int_{\mathbb{R}} \frac{\lambda - i}{\lambda - z} d\sigma_0(\lambda), \quad (21)$$

$$\mathcal{B}(z) = \int_{\mathbb{R}} \frac{\lambda + i}{\lambda - z} d\sigma_0(\lambda), \quad (22)$$

where $\sigma_0(\lambda) = P_+E_0(\lambda)|_{\mathfrak{N}_i}$, $E_0(\lambda)$ is the resolution of identity for H_0 . Then the functions \mathcal{A} and \mathcal{B} satisfy the following identities:

$$\mathcal{A}(z + b) = \frac{z + i}{z + b + i} \mathcal{A}(z), \quad (23)$$

$$\mathcal{B}(z + b) = \frac{z - i}{z + b - i} \mathcal{B}(z). \quad (24)$$

Proof. We prove identity for \mathcal{A} . Identity for \mathcal{B} is proved similarly.

$$\begin{aligned}\mathcal{A}(z + b) &= \int \frac{\lambda - i}{\lambda - z - b} d\sigma_0(\lambda) = \\ &= \frac{1}{z + b + i} \int \left[\frac{1}{\lambda - z - b} - \frac{1}{\lambda + i} \right] (1 + \lambda^2) d\sigma_0(\lambda).\end{aligned}$$

Since (\mathcal{H}, H_0) is the (U, b) -periodic pair, the Weyl-Titchmarsh function $M_{\mathcal{H}, H_0}(z)$ for the pair has period b , from which it follows, that the measure

$d\tau_0(\lambda) = (1 + \lambda^2)d\sigma_0(\lambda)$ also has period b . This condition provides, that

$$\int \left[\frac{1}{\lambda - z - b} - \frac{1}{\lambda + i} \right] d\tau_0(\lambda) = \int \left[\frac{1}{\lambda - z} - \frac{1}{\lambda + i} \right] d\tau_0(\lambda),$$

and the statement regarding the function $\mathcal{A}(z)$ follows.

Corollary 2 . *Let \mathcal{H} be a prime symmetric operator in the Hilbert space \mathfrak{H} with index of defect (m, m) , and H_0 be its orthogonal self-adjoint extension such that the pair (\mathcal{H}, H_0) is a (U, b) -periodic. Then for any other orthogonal self-adjoint extension H of the operator \mathcal{H} corresponding pair (\mathcal{H}, H) is a (U', b) -periodic with some unitary operator U' .*

Proof. In light of the Theorem 1 it is enough to show that periodicity of $M_{\mathcal{H}, H_0}(z)$ implies periodicity of $M_{\mathcal{H}, H}(z)$.

Let σ_0 be the non decreasing operator valued function which provides the integral representation of the $M_{\mathcal{H}, H_0}(z)$. Consider the functional model for the pair (\mathcal{H}, H_0) . Then the domain $\mathfrak{D}(H)$ of the self-adjoint extension H of the operator \mathcal{H} consists of the functions $f(\lambda) \in L^2(\mathbb{R}, \mathfrak{N}_i, d\sigma_0)$ which can be written as

$$f = g + (\varphi_i - V\varphi_{-i}), \quad (25)$$

where $g \in \mathfrak{D}(\mathcal{H})$, that is $\int_{\mathbb{R}} (\lambda + i)g(\lambda)d\sigma_0(\lambda) = 0$, $\varphi \in \mathfrak{N}_i$, $\varphi_{-i} \in \mathfrak{N}_{-i}$, $\|\varphi\| = \|\varphi_{-i}\|$, and V is a some unitary operator in \mathfrak{N}_{-i} . We also have that for $f \in \mathfrak{D}(H)$ $Hf = \mathcal{H}g + i(\varphi_i + V\varphi_{-i})$.

From the definition of Weyl-Titchmarsh function of the pair we have that

$$\frac{M_{\mathcal{H}, H}(z) - M_{\mathcal{H}, H_0}(z)}{1 + z^2} = P_+ [R(z) - R_0(z)]|_{\mathfrak{N}_i},$$

where R and R_0 are resolvents of H and H_0 respectively. Calculating the difference of resolvents, we get the following expression

$$\frac{M_{\mathcal{H}, H}(z) - M_{\mathcal{H}, H_0}(z)}{1 + z^2} = \mathcal{A}(z) (I - V) [(i + z)\mathcal{A}(z)V + (i - z)\mathcal{B}(z)]^{-1} \mathcal{B}(z), \quad (26)$$

where $\mathcal{A}(z)$ and $\mathcal{B}(z)$ are defined by (21) and (22). Using now formulas (23) and (24), we obtain that $M_{\mathcal{H}, H}(z) - M_{\mathcal{H}, H_0}(z) = M_{\mathcal{H}, H}(z + b) - M_{\mathcal{H}, H_0}(z + b)$, and the Corollary is proved.

Let \mathcal{H} be a (U, b) -periodic prime symmetric operator in a Hilbert space \mathfrak{H} with index of defect (m, m) , $(m < \infty)$. Fix orthonormal bases $\{\varphi_j\}_{j=1}^m$ in \mathfrak{N}_i and $\{\psi_j\}_{j=1}^m$ in \mathfrak{N}_{-i} , and a unitary operator V_0 in \mathfrak{N}_{-i} . The matrix of this

operator with respect to the basis $\{\psi_j\}_{j=1}^m$ we also denote by V_0 . Denote by $\mathfrak{D}(H_0)$ the domain of self-adjoint extension H_0 of the operator \mathcal{H} defined as

$$\mathfrak{D}(H_0) = \{f \in \mathfrak{H} | f = f_0 + \sum_j c_j(\varphi_j - V_0\psi_j), f_0 \in \mathfrak{D}(\mathcal{H}), c_j \in \mathbb{C}\}.$$

Since $U\mathfrak{D}(\mathcal{H}) = \mathfrak{D}(\mathcal{H})$ the set $U^n\mathfrak{D}(H_0)$ is the domain of another self-adjoint extension H_n of the operator \mathcal{H} . The extension H_n is defined by the pair of defect subspaces \mathfrak{N}_{i+nb} and \mathfrak{N}_{-i+nb} , and by the unitary operator $V_0^{(n)}$ in the space \mathfrak{N}_{-i+nb} . This operator is defined by the condition that its matrix with respect to the basis $\{U^n\psi_j\}$ coincides with the matrix V_0 . It is easily seen that $V_0^{(n)} = U^n V_0 U^{*n}|_{\mathfrak{N}_{-i+nb}}$.

The extension H_n can be also characterized in terms of the defect subspaces \mathfrak{N}_i and \mathfrak{N}_{-i} and the unitary operator V_n acting on \mathfrak{N}_{-i} . In order to do it it is sufficiently to find the operator V_n from the system of equations

$$\varphi_j - V_n\psi_j = f_{0,j} + \sum_k (U^n\varphi_k - V_0^{(n)}U^n\psi_k)\alpha_{kj}, \quad f_{0,j} \in \mathfrak{D}(\mathcal{H}), j = 1, 2 \dots m.$$

Let us introduce the following $m \times m$ matrices:

$$A_n = [(U^n\psi_k, \psi_l)]_{k,l=1}^m, \quad B_n = [(U^n\varphi_k, \psi_l)], \quad (27)$$

$$C_n = [(U^n\psi_k, \varphi_l)]_{k,l=1}^m, \quad D_n = [(U^n\varphi_k, \varphi_l)]_{k,l=1}^m. \quad (28)$$

Then the matrix of operator V_n with respect to the basis $\{\psi_j\}$ is defined by the expression

$$V_n = T_n(V_0) = -[(nb - 2i)A_n V_0 - nbB_n][nbC_n V_0 - (nb + 2i)D_n]^{-1}. \quad (29)$$

Putting $T_0 = id$ — the identity mapping, we obtain the family $\Gamma = T_n, n \in \mathbb{Z}$ of mappings of the set $U(m)$ of $m \times m$ unitary matrices into itself. By its construction the mappings T_n posses the property $T_n(T_m(\cdot)) = T_{n+m}(\cdot)$. Therefore the family Γ is a group.

From the Corollary 1 we obtain that if for some initial unitary matrix V_0 the trajectory $\{T_k(V_0)\}_{k=-\infty}^{\infty}$ is periodic, that is $T_n(V_0) = V_0$ for some positive integer n , than it is periodic for any other initial matrix with the same period n . In such a situation the operator \mathcal{H} admits (U, nb) — periodic self-adjoint extension, where n is the period of the trajectory of an initial unitary matrix V_0 . We reformulate this property as a property of the group Γ :

Proposition 2 *Let \mathcal{H} be a (U, b) —periodic prime symmetric operator with index of defect (m, m) and Γ be the associated group of mappings of the set $U(m)$ into itself, defined by (27-29). Then the operator \mathcal{H} admits periodic self-adjoint extension if and only if the group Γ is cyclic.*

Examples.

(a) Let $h(\lambda)$ be a nonnegative bounded function which has the period b . Put $d\sigma(\lambda) = h(\lambda)/(1 + \lambda^2)d\lambda$ and use the definition (2). Then the corresponding function has the period b . In particular, for $h(\lambda) = 1 + \sin \lambda$,

$$F(z) = i + e^{iz} - e^{-1}.$$

The function $F(z)$ has the period 2π . It is the Weyl-Titchmarsh function of the pair (\mathcal{H}, H) defined by the formulas (5), (6), (7).

(b) Let $\mathfrak{H} = L_m^2[0, l]$, and the operator \mathcal{H} is defined as following:

Its domain is the set of all absolutely continuous functions $f(t) = \{f_k(t)\}_{k=1}^m \in \mathfrak{H}$, such that $f' \in \mathfrak{H}$, $f(0) = f(l) = 0$;

$$\mathcal{H}f(t) = i \frac{df}{dt}. \quad (30)$$

The operator \mathcal{H} has defect index (m, m) . The defect subspace \mathfrak{N}_i is generated by the columns of the $m \times m$ matrix $\exp(t)I_m$. There is one-to-one correspondence between set of self-adjoint extensions of \mathcal{H} and $m \times m$ unitary matrices V . Any self-adjoint extension H_V of \mathcal{H} is obtained as follows:

Its domain is set of all absolutely continuous functions f from $L_m^2[0, l]$, such that $f' \in L_m^2[0, 1]$, and $f(0) = Vf(l)$, where V is a unitary matrix in \mathbb{C}^m . For the pair (\mathcal{H}, H_V) the Weyl-Titchmarsh function $M_{\mathcal{H}, H_V}$ is equal to

$$M_{\mathcal{H}, H_V}(z) = -iI_m + \frac{2i}{e^{2l} - 1} (e^{l(1-iz)} - 1)(I_m - e^{-izl}V)^{-1}(I_m - e^lV). \quad (31)$$

This function has the period $2\pi/l$. Therefore, the operators (30) and H form a $2\pi/l$ -periodic pair and the same is true for any other self-adjoint extension of (30). The unitary operator U , such that $U\mathcal{H}U^* = \mathcal{H} - (2\pi/l)I$, and similar equality for H is the operator of multiplication by $\exp(-2\pi it/l)$.

(c) More generally, consider the operator $\mathcal{H}_1 = id/dt + V(t)$ on $L_m^2[0, 1]$ with the same domain that above. V is a hermitian, bounded measurable matrix function which satisfies condition $V(0) = V(l)$. Then the operator \mathcal{H}_1 is symmetric with index of defect (m, m) . Let H_1 be its self-adjoint extension. Then the Weyl-Titchmarsh function $M_{\mathcal{H}_1, H_1}(z)$ has the period $2\pi/l$.

According to well-known theorem by M.Livsic [7] a prime symmetric operator with index of defect $(1, 1)$ which admits a quasi-hermitian extension smo_v

without spectrum in the finite complex plane is unitarily equivalent to operator, described in example (b) with $m = 1$ for $l = 2\text{tr}(\Im \mathcal{H}_v^{-1}) > 0$. Therefore, we have the following statement.

Theorem 4 *Let \mathcal{H} be a prime symmetric operator with index of defect $(1, 1)$, and H be a self-adjoint extension of \mathcal{H} . Suppose that \mathcal{H} admits quasi-self-adjoint extension \mathcal{H}_v without spectrum. Then the Weyl-Titchmarsh function $M_{\mathcal{H}, H}(z)$ of the pair (\mathcal{H}, H) is a periodic one. Its period is equal to $\pi/\text{tr}(\Im \mathcal{H}_v^{-1})$.*

This theorem does not admit generalization for the case of larger defect numbers. Indeed, let $\mathfrak{H} = L^2[0, l]$, and let $0 < \xi < l$. Consider the symmetric operator \mathcal{H} on \mathfrak{H} , defined as following:

The domain $\mathfrak{D}(\mathcal{H})$ is the set of all functions $f(t)$ which are absolutely continuous for $0 < t < \xi$ and $\xi < t < l$, $f' \in \mathfrak{H}$, and $f(0) = f(\xi) = f(l) = 0$. For $f \in \mathfrak{D}(\mathcal{H})$ $\mathcal{H}f = idf/dt$. The index of defect of \mathcal{H} is equal $(2, 2)$. This operator admits quasi-self-adjoint extension \mathcal{H}_v without spectrum, and \mathcal{H}_v^{-1} is dissipative and unicellular [2]. The operator \mathcal{H} is isomorphic to the direct sum $\mathcal{H}_1 \oplus \mathcal{H}_2$ of two first order differential operators with zero boundary conditions on $[0, \xi]$ and $[\xi, l]$ respectively. Let H be the self-adjoint extension of $\mathcal{H}_1 \oplus \mathcal{H}_2$ obtained by imposing the following conditions: $f(0) = \omega_1 f(\xi - 0)$, $f(\xi + 0) = \omega_2 f(l)$, where $|\omega_1| = |\omega_2| = 1$. The the Weyl-Titchmarsh function $M_{\mathcal{H}, H}(z)$ of the pair (\mathcal{H}, H) is a 2×2 diagonal matrix

$$M_{\mathcal{H}, H}(z) = \begin{bmatrix} M_1(z) & 0 \\ 0 & M_2(z) \end{bmatrix},$$

where

$$\begin{aligned} M_1(z) &= -i + 2i(e^{\xi(1-iz)} - 1)(1 - \omega_1 e^{\xi}) / [(e^{2\xi} - 1)(1 - \omega_1 e^{-iz\xi})], \\ M_2(z) &= -i + 2i(\omega_2 e^l - e^{\xi})(e^l e^{-(l-\xi)iz} - e^{\xi}) / [(e^{2l} - e^{2\xi})(\omega_2 - e^{-iz(l-\xi)})]. \end{aligned}$$

M_1 has the period $2\pi/\xi$, function M_2 has the period $2\pi/(l - \xi)$. Therefore, if $\xi/(l - \xi)$ is an irrational number, the function $M_{\mathcal{H}, H}$ is not a periodic.

4 Operators With Constant Weyl-Titchmarsh Function.

Let H be a self-adjoint operator, and let $W(t), t \in \mathbb{R}$ be the one-parametric group of unitary operators generated by H ($W(t) = \exp(iHt)$). If H is a (U, b) -periodic operator, then the following commutative relation is fulfilled:

$$UW(t) = e^{-itb}W(t)U. \quad (32)$$

So far we have considered the Weyl-Titchmarsh functions, which are invariant under some fixed shift b of the argument. Let $F(z)$ be a function whose values are operators on m -dimensional space \mathfrak{N} , which admits representation (13) and invariant under arbitrary real shift, that is $F(z+s) = F(z)$ for any real s . In such a situation the function $F(z)$ is, of course, constant in each half-plane,

$$F(z) = \begin{cases} iI_{\mathfrak{N}} & z \in \mathbb{C}_+, \\ -iI_{\mathfrak{N}} & z \in \mathbb{C}_-. \end{cases} \quad (33)$$

These properties are fulfilled if and only if $d\tau(\lambda) = \pi^{-1}d\lambda I_{\mathfrak{N}}$.

We have $F(z) = M_{\tilde{\mathcal{H}}, \tilde{H}}(z)$ for the pair $(\tilde{\mathcal{H}}, \tilde{H})$ acting in the Hilbert space $\tilde{\mathfrak{H}} = L^2(\mathbb{R}, \mathfrak{N}, \pi^{-1}d\lambda)$, where

$$\mathfrak{D}(\tilde{H}) = \{f \in L^2(\mathbb{R}, \mathfrak{N}, \pi^{-1}d\lambda) \mid \int_{\mathbb{R}} (1 + \lambda^2) \|f(\lambda)\|_{\mathfrak{N}}^2 d\lambda < \infty\}; \quad (34)$$

$$(\tilde{H}f)(\lambda) = \lambda f(\lambda); \quad (35)$$

$$\mathfrak{D}(\tilde{\mathcal{H}}) = \{f \in \mathfrak{D}(H) \mid \int_{\mathbb{R}} f(\lambda) d\lambda = 0\}; \quad (36)$$

$$(\tilde{\mathcal{H}}f)(\lambda) = \lambda f(\lambda). \quad (37)$$

According to the Theorem 3 for any real number s there is a unitary operator $\tilde{V}(s)$ on $L^2(\mathbb{R}, \mathfrak{N}, \pi^{-1}d\lambda)$ such that $\tilde{V}(s)\tilde{H}\tilde{V}^*(s) = \tilde{H} - sI$, and $\tilde{V}(s)\tilde{\mathcal{H}}\tilde{V}^*(s) = \tilde{\mathcal{H}} - sI$. The operators $\tilde{V}(s)$ act as following: $(\tilde{V}(s)f)(\lambda) = f(\lambda - s)$. Therefore, the family $\{\tilde{V}(s)\}$ is strongly continuous unitary group. If $\tilde{W}(t) = \exp(it\tilde{H})$, then

$$\tilde{V}(s)\tilde{W}(t) = e^{-ist}\tilde{W}(t)\tilde{V}(s), \quad (38)$$

which is the Weyl's form of the canonical commutative relation.

Theorem 5 . *Let \mathcal{H} be a prime symmetric operator with index of defect (m, m) , $m < \infty$, $H \supset \mathcal{H}$ be its self-adjoint extension, and let $W(t) (= \exp(itH))$ be the unitary group generated by H . Then the following conditions are equivalent*

- (1) *There exists a unitary group $V(s)$ of s -type operators such that $V(s)W(t) = e^{-its}W(t)V(s)$;*
- (2) *The Weyl-Titchmarsh function $M_{\mathcal{H}, H}(z) = iI_{\mathfrak{N}_i}$ for $z \in \mathbb{C}_+$, and $M_{\mathcal{H}, H}(z) = -iI_{\mathfrak{N}_i}$ for $z \in \mathbb{C}_-$, where \mathfrak{N}_i , $\dim \mathfrak{N}_i = m$, is the defect subspace of \mathcal{H} .*

Let G be the self-adjoint operator such that $V(s) = \exp(isG)$. Then condition 1 means that

$$[G, H] = iI$$

on a dense subset of \mathfrak{H} .

Proof. We have proved that from the statement 2 follows the statement 1. Let the statement 1 is fulfilled. Then for $f \in \mathfrak{D}(H)$ it follows that $V(s)f \in \mathfrak{D}(H)$ for any $s \in \mathbb{R}$, and $V(s)Hf = (H - sI)V(s)f$. It is not hard to show that last condition along with the assumption about special structure of operators $V(s)$ implies that the operator \mathcal{H} is also (U, s) -periodic for any real s . Therefore the Weyl-Titchmarsh function of the pair (\mathcal{H}, H) is constant in upper half-plane and in lower half-plane.

The Theorem is proved.

The pair (\mathcal{H}, H) is unitarily equivalent to its functional model $(\tilde{\mathcal{H}}, \tilde{H})$ given by the formulas (34-37). In such representation the group $\tilde{V}(s)$, as it was pointed out above, can be selected as group of shifts, $(\tilde{V}(s)f)(\lambda) = f(\lambda - s)$.

Consider the case $m = 1$. The group $\tilde{W}(t) (= \exp(i\tilde{H}t))$ is the group of multiplication by $\exp(i\lambda t)$ in the space $\mathfrak{H} = L^2(\mathbb{R}, \pi^{-1}d\lambda)$, and $(\tilde{V}(s)f)(\lambda) = f(\lambda - s)$. This statement follows from the fact that for each s the operator $V(s)$ satisfies $\tilde{V}(s)H = (\tilde{H} - sI)\tilde{V}(s)$, Proposition 1, and the group property $(\tilde{V}(s_1 + s_2) = \tilde{V}(s_1)\tilde{V}(s_2))$. Therefore, we obtained the statement of the Stone-von Neumann theorem for degree of freedom 1 ([8]).

Let D be the selfadjoint operator, such that $\tilde{V}(s) = \exp(iDs)$. Then

$$\mathfrak{D}(D) = \{f \in L^2(\mathbb{R}, \pi^{-1}d\lambda) | f \in AC(-\infty, \infty); f' \in L^2(\mathbb{R}, \pi^{-1}d\lambda)\}, \quad (39)$$

$$(Df)(\lambda) = if'(\lambda). \quad (40)$$

The operator D is the selfadjoint extension of the operator \mathcal{D} defined as

$$\mathfrak{D}(\mathcal{D}) = \{f \in L^2(\mathbb{R}, \pi^{-1}d\lambda) | f \in AC(-\infty, \infty); f' \in L^2(\mathbb{R}, \pi^{-1}d\lambda); f(0) = 0\}, \quad (41)$$

$$(\mathcal{D}f)(\lambda) = if'(\lambda). \quad (42)$$

Again applying the Theorem 5, we obtain that the Weyl-Titchmarsh function of the pair (\mathcal{D}, D) is constant (this fact can be checked, of course, by direct calculations.). If D_ω and \tilde{H}_θ be arbitrary selfadjoint extensions of \mathcal{D} and $\tilde{\mathcal{H}}$ respectively, then, according to the Corollary 1, the Weyl-Titchmarsh functions $M_{\tilde{\mathcal{H}}, \tilde{H}_\theta}(z)$ and $M_{\mathcal{D}, D_\omega}(z)$ are constant. Therefore pair $(\tilde{\mathcal{H}}, \tilde{H}_\theta)$ is unitarily equivalent to the pair $(\tilde{\mathcal{H}}, \tilde{H})$, and pair (\mathcal{D}, D_ω) is unitarily equivalent to the pair (\mathcal{D}, D) .

We have

$$\mathfrak{D}(\tilde{H}_\theta) = \{f | f(\lambda) = f_0 + (\frac{1}{\lambda - i} - \frac{\theta}{\lambda + i})z\}, \quad (43)$$

where $f_0 \in \mathfrak{D}(\mathcal{H})$, $|\theta| = 1$, and $z \in \mathbb{C}$.

$$(\tilde{H}_\theta f)(\lambda) = \lambda f_0(\lambda) + i[1/(\lambda - i) + \theta/(\lambda + i)]z, \quad (44)$$

and $\tilde{H} = \tilde{H}_1$. The unitary operator Γ_θ such that $\tilde{H}_\theta = \Gamma_\theta \tilde{H}_1 \Gamma_\theta^*$ acts as following: $(\Gamma_\theta f)(\lambda) = \theta \hat{f}_+(\lambda) + \hat{f}_-(\lambda)$, where $f = \hat{f}_+ + \hat{f}_-$ is the (unique) representation of function $f \in L^2(\mathbb{R}, d\lambda)$ as the sum of functions $\hat{f}_+ \in H_+^2$ and $\hat{f}_- \in H_-^2$. Since $1/(\lambda - i) \in H_-^2$, and $1/(\lambda + i) \in H_+^2$, we need to show that $\Gamma_\theta \mathfrak{D}(\mathcal{H}) \subset \mathfrak{D}(\mathcal{H})$. For $f \in \mathfrak{D}(\mathcal{H})$ we have

$$f(\lambda) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\lambda t} F(t) dt,$$

where $F \in L^2(\mathbb{R}, dt)$, $F' \in L^2(\mathbb{R}, dt)$, and $F(0) = 0$.

$$(\Gamma_\theta f)(\lambda) \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\lambda t} F(t) [\theta \chi_+(t) + \chi_-(t)] dt, \quad (45)$$

where χ_\pm are indicators functions of the positive and negative semaxes respectively. The integrand of the last expression is equal to zero at $t = 0$, therefore $\Gamma_\theta f \in \mathfrak{D}(\mathcal{H})$. It is also clear that $\widehat{(\mathcal{H}f)}_\pm = \lambda \hat{f}_\pm$, and $\Gamma_\theta^* = \Gamma_{\bar{\theta}}$.

For the operator D_ω we have

$$\begin{aligned} \mathfrak{D}(D_\omega) = \{f \in L^2(\mathbb{R}, d\lambda) | f \in AC([-R, 0]) \cap AC([0, R]) \forall R > 0; \\ f(0_-) = \omega f(0_+), |\omega| = 1; f' \in L^2(\mathbb{R}, d\lambda)\} \end{aligned} \quad (46)$$

$$(D_\omega f)(\lambda) = i f'(\lambda), \quad (47)$$

and $D = D_1$.

The unitary operator J_ω such that $D_\omega = J_\omega D_1 J_\omega^*$ acts as following:

$$(J_\omega f)(\lambda) = [\chi_-(\lambda) + \omega \chi_+(\lambda)] f(\lambda), \quad (48)$$

$$J_\omega^* = J_{\bar{\omega}}.$$

From (45) and (48) it follows that $\Gamma_\theta J_\omega = J_\omega \Gamma_\theta$.

Let \tilde{W}_θ be the unitary group generated by \tilde{H}_θ , and $\tilde{V}_\omega(s)$ be the unitary group generated by D_ω . It is not hard to describe their actions. For example, the group $\tilde{V}_\omega(s)$ acts as following:

for $s > 0$

$$(\tilde{V}_\omega(s)f)(\lambda) = \begin{cases} f_-(\lambda - s) & \lambda < 0 \\ \omega f_-(\lambda - s) & 0 \leq \lambda \leq s \\ f_+(\lambda - s) & \lambda \geq s \end{cases}$$

and for $s < 0$

$$(\tilde{V}_\omega(s)f)(\lambda) = \begin{cases} f_-(\lambda - s) & \lambda < s \\ \bar{\omega} f_+(\lambda - s) & s \leq \lambda < 0 \\ f_+(\lambda - s) & \lambda \geq 0 \end{cases}$$

It is clear, that $\Gamma_\theta D_1 = D_1 \Gamma_\theta$, and $J_\omega H_1 = H_1 J_\omega$.

Proposition 3 *Let \tilde{H}_θ and D_ω be the operators defined by (43-44) and (46-47) respectively. Then for the unitary groups $\tilde{W}_\theta(t)$ and $\tilde{V}_\omega(s)$ generated by \tilde{H}_θ and D_ω respectively the H. Weyl commutative relation (38) is fulfilled, that is*

$$\tilde{V}_\omega(s)\tilde{W}_\theta(t) = e^{-its}\tilde{W}_\theta(t)\tilde{V}_\omega(s)$$

The proposition follows from the following chain of equalities where above mentioned properties of the operators Γ_θ , J_ω , D_1 , and \tilde{H}_1 are used:

$$\begin{aligned} \tilde{V}_\omega(s)\tilde{W}_\theta(t) &= J_\omega \tilde{V}_1(s) J_\omega^* \Gamma_\theta \tilde{W}_1(t) \Gamma_\theta^* = J_\omega \Gamma_\theta \tilde{V}_1(s) \tilde{W}_1(t) \Gamma_\theta^* J_\omega^* = \\ &= e^{-ist} J_\omega \Gamma_\theta \tilde{W}_1(t) \tilde{V}_1(s) \Gamma_\theta^* J_\omega^* = e^{-ist} \Gamma_\theta \tilde{W}_1(t) \Gamma_\theta^* J_\omega \tilde{V}_1(s) J_\omega^* \\ &= e^{-ist} \tilde{W}_\theta(t) \tilde{V}_\omega(s). \end{aligned}$$

Last proposition admits reformulation in abstract form.

Proposition 4 . *Let F_1 and G_1 be self-adjoint operators with simple spectra acting in a Hilbert space \mathfrak{H} , and corresponding unitary groups $V_1(s)(= \exp(iF_1 s))$ and $W_1(t)(= \exp(iG_1 t))$ satisfy (38). Then:*

- (1) *There are prime symmetric operators F_0 and G_0 which have index of defect $(1, 1)$ such that $F_0 \subset F_1$ and $G_0 \subset G_1$;*
- (2) *For any other self-adjoint extensions F_ω and G_θ of the operators F_0 and G_0 respectively the corresponding unitary groups $V_\omega(s)$ and $W_\theta(t)$ also satisfy (38);*
- (3) *There exists the unitary operator $U_{\theta\omega} : \mathfrak{H} \rightarrow L^2(\mathbb{R}, \pi^{-1} d\lambda)$ such that $F_\omega = U_{\theta\omega}^* D_\omega U_{\theta\omega}$, $G_\theta = U_{\theta\omega}^* \tilde{H}_\theta U_{\theta\omega}$, $F_0 = U_{\theta\omega}^* \mathcal{D} U_{\theta\omega}$, and $G_0 = U_{\theta\omega}^* \tilde{\mathcal{H}} U_{\theta\omega}$.*

This proposition follows from the Stone-Von Neumann Theorem and previous consideration. It also gives some refinement of the Stone-von Neumann's

Theorem. The case $\omega = \theta = 1$ is the most well-known. It corresponds to the operators of momentum and coordinate in quantum mechanics.

Consider one more example of the pair with constant Weyl-Titchmarsh function. Let $\mathfrak{H} = L^2(\mathbb{R}, dt)$ and the self-adjoint operator is defined by the differential expression

$$Lf = -\frac{1}{\gamma} \frac{d^2 f}{dx^2} + xf, \quad (49)$$

where γ is a real constant. Corresponding self-adjoint operator describes the particle in uniform electrical field. This operator via Fourier transform is unitarily equivalent to the self-adjoint operator H defined as

$$\begin{aligned} (Hf)(t) &= i \frac{df}{dt} + \frac{1}{\gamma} t^2 f(t); \\ \mathfrak{D}(H) &= \{f \in L^2(\mathbb{R}, dt) | f \in AC(-\infty, \infty), f' \in L^2(\mathbb{R}, dt), \\ &\quad t^2 f(t) \in L^2(\mathbb{R}, dt)\}. \end{aligned}$$

Define the operator \mathcal{H} as following

$$\begin{aligned} \mathfrak{D}(\mathcal{H}) &= \{f \in L^2(\mathbb{R}, dt) | f \in AC(-\infty, 0] \cup [0, \infty), f(0) = 0, f' \in L^2(\mathbb{R}, dt), \\ &\quad t^2 f(t) \in L^2(\mathbb{R}, dt)\}; \end{aligned}$$

$$(\mathcal{H}f)(t) = i \frac{df}{dt} + \frac{1}{\gamma} t^2 f(t).$$

The operator \mathcal{H} is a symmetric operator with index of defect $(1, 1)$, and H is the selfadjoint extension of \mathcal{H} . For any real s define a unitary operator U_s on \mathfrak{H} by $(U_s f)(t) = e^{ist} f(t)$. Then we have $U_s \mathfrak{D}(\mathcal{H}) = \mathfrak{D}(\mathcal{H})$, $U_s \mathfrak{D}(H) = \mathfrak{D}(H)$, and $U_s H U_s^* = (H - sI)$ that is the pair (\mathcal{H}, H) is (U_s, s) -periodic. From the Theorem 5 it follows now that the Weyl-Titchmarsh function of the pair (\mathcal{H}, H) is constant in each half-plane. Therefore, the operator H is unitarily equivalent to the operator of multiplication in $L^2(\mathbb{R}, dt)$.

Thus, the self-adjoint operator, generated by the differential expression (49) and its appropriate symmetric restriction have the constant Weyl-Titchmarsh function.

Let V be a bounded, measurable, periodic, real valued periodic function. Without loss of generality we assume that the period of V is 2π . The Fourier series of V

$$\sum_{k=-\infty}^{\infty} \hat{V}(k) e^{ikx}$$

converges to $V(x)$ a.e., where $\hat{V}(k)$ are the Fourier coefficients of the function V .

Consider the self-adjoint operator

$$L_1 = L + V.$$

Again Fourier transform gives that the operator L_1 is unitarily equivalent to the operator

$$H_1 f = i \frac{df}{dt} + \frac{1}{\gamma} t^2 f + \sum_k \hat{V}(k) f(t+k).$$

Operator H_1 is the selfadjoint extension of the symmetric operator \mathcal{H}_1 with the same domain that the operator \mathcal{H} above. Now we have

$$U_s H_1 f - H_1 U_s f = -s e^{ist} f + e^{ist} \sum_k \hat{V}(k) (1 - e^{isk}) f(t+k),$$

and similar expression for $U_s \mathcal{H}_1 - \mathcal{H}_1 U_s$. Putting $s = 2\pi$, we see that $U_{2\pi} H_1 - H_1 U_{2\pi} = -2\pi U_{2\pi}$, and similar equation for \mathcal{H}_1 . Therefore, the pair (\mathcal{H}_1, H_1) is 2π -periodic. Thus the pair (\mathcal{L}_1, L_1) where \mathcal{L}_1 is the symmetric restriction of the Shrödinger operator L_1 with index of defect $(1, 1)$ (inverse Fourier Transform of \mathcal{H}_1) has the 2π -periodic Weyl-Titchmarsh function.

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